ASYMPTOTIC BEHAVIOUR OF A PHASE FIELD MODEL WITH THREE COUPLED EQUATIONS WITHOUT UNIQUENESS

PEDRO MARÍN-RUBIO, GABRIELA PLANAS, AND JOSÉ REAL

Abstract. We prove the existence of weak solutions for a phase field model with three coupled equations with unknown uniqueness, and state several dynamical systems depending on the regularity of the initial data. Then, the existence of families of global attractors (level-set depending) for the corresponding multi-valued semiflows is established, applying an energy method. Finally, using the regularizing effect of the problem, we prove that these attractors are in fact the same.

1. Introduction and statement of the problem

The phase-field method provides a mathematical description for free-boundary problems associated to physical processes with phase transitions. In this methodology each phase is distinguished by a so called phase field. In different phases the phase field attains different values and interfaces are modelled by a diffuse interface. There exists a wide literature devoted to several modelling, among other papers we may cite [3, 13, 5, 19].

An interesting aspect of the problem, jointly with the well-posedness of each model, concerns the asymptotic behaviour of the system, since this analysis can provide useful information on the future evolution of the dynamic. In particular, the global attractor has been proved an extremely useful tool in the study of the asymptotic behaviour in many physical situations (e.g. cf. [21, 20, 18, 11]).

The long time behaviour of solutions to phase-field models for pure materials has been investigated by many authors, for instance see [7, 2, 10, 12, 17, 8, 22].

Besides the above phase-field systems, that are only concerned with one material and consisting of two coupled equations, for the case of binary alloys, a new model was proposed in [4]. It needs to contain a new variable to indicate the fraction of one of the two materials in the mixture. This finally yields to a highly nonlinear parabolic system of three partial differential equations with three independent variables: phase-field, solute concentration, and temperature, which recently was analyzed rigourously from the mathematical point of view in [1].

Namely, the system, which we are interested in studying its asymptotic behaviour, is the following:

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$$\alpha\varepsilon^2\phi_t - \varepsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta(\theta - c\theta_A - (1-c)\theta_B) \quad \text{in} \quad Q, \tag{1}$$

$$C_V\theta_t + \frac{l}{2}\phi_t = \nabla\cdot[K_1(\phi)\nabla\theta] \quad \text{in} \quad Q, \tag{2}$$

$$c_t = K_2(\Delta c + M\nabla\cdot[c(1-c)\nabla\phi]) \quad \text{in} \quad Q, \tag{3}$$

$$0 \le c \le 1 \quad \text{in} \quad Q, \tag{4}$$

$$\frac{\partial\phi}{\partial n} = 0, \ \frac{\partial\theta}{\partial n} = 0, \ \frac{\partial c}{\partial n} = 0 \quad \text{on} \quad \Sigma, \tag{5}$$

where $Q = \Omega\times(0,+\infty)$ and $\Sigma = \partial\Omega\times(0,+\infty)$, being Ω an open connected bounded domain of $\mathbb{R}^N$ with $N = 2$ or 3, with smooth boundary $\partial\Omega$. The order parameter (phase-field) ϕ is the state variable characterizing the different phases; the function θ represents the temperature; the concentration $c \in [0,1]$ denotes the fraction of one of the two materials in the mixture. The parameter $\alpha > 0$ is the relaxation scaling; the parameter β is given by $\beta = \varepsilon[s]/3\sigma$, where $\varepsilon > 0$ is a measure of the interface width, σ the surface tension and $[s]$ the entropy density difference between phases; $C_V > 0$ is the specific heat; the constant $l > 0$ the latent heat; θ_A and θ_B are the respective melting temperatures of each of the two materials in the alloy; $K_2 > 0$ is the solute diffusivity; M is a constant related to the slopes of solidus and liquidus lines; K_1 denotes the thermal conductiviy. This physical parameter is assumed, as in [13], to be a function depending on the order parameter ϕ. More precisely, throughout this paper we assume that K_1 is a (globally) Lipschitz function and there exist positive constants $\underline{k}_1$, $\overline{k}_1$ such that

$$0 < \underline{k}_1 \le K_1(r) \le \overline{k}_1 \quad \text{for all } r \in \mathbb{R}. \tag{6}$$

Concerning the nonlinearity $\phi - \phi^3$ we point out that other nonlinearities can be treated with a little more work (cf. [2]).

As far as uniqueness of solution is unknown for (1)–(5), we must use multi-valued dynamical systems for our approach. In this sense we are close to [8], although there the boundary conditions were Dirichlet and ours are Neumann. This point seems to represent the situation of a phase-field problem in a more realistic way. However, this involves additional mathematical duties, as long as a Poincaré inequality cannot be applied directly as in the Dirichlet case, but the Poincaré-Wirtinger inequality for $H^1(\Omega)$, making the most of a characterization of the average of the solutions and an invariance property. One clear and distinguishing consequence of this framework is that the study of the attractors must be performed in, say, a "level set" sense instead of the whole space, which is similar to that in [2].

The structure of the paper is as follows: in Section 2 we establish a result on existence of weak solutions and a regularizing effect for problem (1)–(5) with initial data in suitable metric spaces, roughly speaking $(L^2)^3$ and $H^1 \times (L^2)^2$. This is a slight improvement of the result in [1], necessary for the statement of dynamical systems. In Section 3 we recall briefly some basic facts from multi-valued analysis that will be used for the study of the asymptotic behaviour in problems where there is no uniqueness or it is unknown. In particular, necessary and sufficient results concerning with the existence of global attractors are given. The construction of several suitable multi-valued semiflows and estimates on the solutions leading to the existence of absorbing sets are given in Section 4. Finally, compact and continuity properties are analyzed in Section 5 to conclude the existence of a family of global attractors in several phase-spaces. A complete answer about the relationship of all these sets is given at the end of the paper.

2. Existence of solutions

Let us firstly introduce some notation which will be used hereafter all through the paper.

For a given metric space $(\mathcal{X}, d)$, $P(\mathcal{X})$, $B(\mathcal{X})$, $C(\mathcal{X})$, and $K(\mathcal{X})$ will denote the class of all nonempty, nonempty and bounded, nonempty and closed, and nonempty and compact subsets of $\mathcal{X}$ respectively. In addition, denote the Hausdorff semidistance in $P(\mathcal{X})$ by

$$\mathrm{dist}_{\mathcal{X}}(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b)$$

for any subsets A, $B \in P(\mathcal{X})$. We will denote $(\cdot,\cdot)$ and $|\cdot|$ the inner product and its associated norm in $L^2(\Omega)$ or in $L^2(\Omega)^N$, and we will use $((\cdot,\cdot))$ and $\|\cdot\|$ to denote the inner product and its associated norm in $H^1(\Omega)$, where

$$((u, v)) = (u, v) + (\nabla u, \nabla v), \quad u, v \in H^1(\Omega).$$

Otherwise, the norm in other spaces will be fully specified. The duality product between $H^1(\Omega)'$ and $H^1(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$.

Just for the sake of clarity in the reading, when convenient, sequences in the paper will be denoted by an upper-script n or μ, instead of (n) or (μ). No confusion arises with any power of a value, since the only power used in the paper is for ϕ^3, and for sequences will denoted by $(\phi^n)^3$.

In this section we establish existence of solutions for an initial value problem associated with (1)–(5) in a suitable sense that will enable us to define several multi-valued semiflows for the problem.

Theorem 1. *With the above notation, let be given $(\phi_0, \theta_0) \in (L^2(\Omega))^2$, and $c_0 \in L^2(\Omega; [0, 1])$, i.e. $c_0 \in L^2(\Omega)$ such that $0 \le c_0(x) \le 1$ a.e. in Ω. Then, under assumption (6), there exist functions ϕ, θ, $c : Q \to \mathbb{R}$, such that for any $T > 0$,*

(i) $\phi \in L^2(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega)) \cap L^4(0, T; L^4(\Omega))$, $\phi_t \in L^2(0, T; H^1(\Omega)') + L^{4/3}(0, T; L^{4/3}(\Omega))$, $\phi(0) = \phi_0$,

(ii) $\theta \in L^2(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega))$, $\theta_t \in L^2(0, T; H^1(\Omega)') + L^{4/3}(0, T; L^{4/3}(\Omega))$, $\theta(0) = \theta_0$,

(iii) $C_V \theta_t + \dfrac{l}{2}\phi_t \in L^2(0, T; H^1(\Omega)')$,

(iv) $c \in L^2(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega))$, $c_t \in L^2(0, T; H^1(\Omega)')$, $c(0) = c_0$, $0 \le c \le 1$ *a.e. in* Q,

and satisfy the equations

$$\begin{aligned} &\alpha\varepsilon^2 \int_0^T \langle \phi_t(t), \eta(t)\rangle\, dt + \varepsilon^2 \int_0^T (\nabla\phi(t), \nabla\eta(t))\, dt \\ = \;& \frac{1}{2}\int_0^T (\phi(t) - \phi^3(t), \eta(t))\, dt \\ &+\beta \int_0^T (\theta(t) + (\theta_B - \theta_A)c(t) - \theta_B, \eta(t))\, dt, \end{aligned} \tag{7}$$

for any $\eta \in L^2(0,T;H^1(\Omega)) \cap L^4(0,T;L^4(\Omega))$,

$$\begin{aligned} \int_0^T \langle C_V \theta_t(t) + \frac{l}{2}\phi_t(t), \eta(t)\rangle \, dt & \\ + \int_0^T (K_1(\phi(t))\nabla\theta(t), \nabla\eta(t))\, dt &= 0, \end{aligned} \tag{8}$$

for any $\eta \in L^2(0,T;H^1(\Omega))$, *and*

$$\begin{aligned} \int_0^T \langle c_t(t), \eta(t)\rangle \, dt + K_2 \int_0^T (\nabla c(t), \nabla\eta(t))\, dt & \\ + K_2 M \int_0^T (c(t)(1-c(t))\nabla\phi(t), \nabla\eta(t))\, dt &= 0, \end{aligned} \tag{9}$$

for any $\eta \in L^2(0,T;H^1(\Omega))$.

If in addition $\phi_0 \in H^1(\Omega)$, *then, for any solution* (ϕ,θ,c), *one has that*

$$\phi \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega)), \quad \phi_t \in L^2(0,T;L^2(\Omega)), \ \forall T>0,$$

$\dfrac{\partial\phi}{\partial\nu} = 0$ *a.e. on* Σ, *and* ϕ *satisfies (1) a.e. in* Q.

Proof. We consider two sequences $\{\phi_0^n\}_{n\geq1} \subset H^2(\Omega)$ and $\{c_0^n\}_{n\geq1} \subset C^1(\overline{\Omega})$ such that $\dfrac{\partial\phi_0^n}{\partial\nu} = 0$ on $\partial\Omega$, $0 \leq c_0^n \leq 1$ a.e. in $\overline{\Omega}$, $\phi_0^n \to \phi_0$ in $L^2(\Omega)$ and $c_0^n \to c_0$ in $L^2(\Omega)$ as $n \to +\infty$.

Our starting point is Theorem 1 in [1] (see also [1, Remark 2 in p.1192]). From this theorem we know that for any fixed $T>0$ and for each $n \geq 1$ there exist functions ϕ^n, θ^n, and c^n, defined on $\Omega\times(0,T)$, such that

- (i') $\phi^n \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega))$, $\phi_t^n \in L^2(0,T;L^2(\Omega))$, $\dfrac{\partial\phi^n}{\partial\nu} = 0$ a.e. on $\partial\Omega\times(0,T)$, and $\phi^n(0) = \phi_0^n$,
- (ii') $\theta^n \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $\theta_t^n \in L^2(0,T;H^1(\Omega)')$, and $\theta^n(0) = \theta_0$,
- (iii') $c^n \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $c_t^n \in L^2(0,T;H^1(\Omega)')$, $c^n(0) = c_0^n$, and $0 \leq c^n \leq 1$ a.e. in $\Omega\times(0,T)$,

and satisfy the equations

$$\alpha\varepsilon^2\phi_t^n - \varepsilon^2\Delta\phi^n = \frac{1}{2}(\phi^n - (\phi^n)^3) + \beta(\theta^n + (\theta_B - \theta_A)c^n - \theta_B) \quad \text{a.e. in } \Omega\times(0,T), \tag{10}$$

$$\begin{aligned} C_V \int_0^T \langle \theta_s^n(s), \eta(s)\rangle \, ds + \frac{l}{2}\int_0^T (\phi_s^n(s), \eta(s))\, ds & \\ + \int_0^T (K_1(\phi^n(s))\nabla\theta^n(s), \nabla\eta(s))\, ds &= 0, \end{aligned} \tag{11}$$

$$\begin{aligned} \int_0^T \langle c_s^n(s), \eta(s)\rangle \, ds + K_2 \int_0^T (\nabla c^n(s), \nabla\eta(s))\, ds & \\ + K_2 M \int_0^T (c^n(s)(1-c^n(s))\nabla\phi^n(s), \nabla\eta(s))\, ds &= 0, \end{aligned} \tag{12}$$

for any $\eta \in L^2(0,T;H^1(\Omega))$.

We introduce the auxiliary functions u^n defined by

$$u^n = C_V\theta^n + \frac{l}{2}\phi^n, \quad n \geq 1.$$

Then, from (10)-(12) we obtain

$$\begin{aligned} &\alpha\varepsilon^2\int_0^T \langle\phi_s^n(s),\eta(s)\rangle\,ds + \varepsilon^2\int_0^T(\nabla\phi^n(s),\nabla\eta(s))\,ds \\ = \;& \frac{1}{2}\int_0^T(\phi^n(s)-(\phi^n)^3(s),\eta(s))\,ds \\ &+\beta\int_0^T(\frac{1}{C_V}u^n(s)-\frac{l}{2C_V}\phi^n(s)+(\theta_B-\theta_A)c^n(s)-\theta_B,\eta(s))\,ds, \end{aligned} \tag{13}$$

$$\begin{aligned} &\int_0^T\langle u_s^n(s),\eta(s)\rangle\,ds + \frac{1}{C_V}\int_0^T(K_1(\phi^n(s))\nabla u^n(s),\nabla\eta(s))\,ds \\ = \;& \frac{l}{2C_V}\int_0^T(K_1(\phi^n(s))\nabla\phi^n(s),\nabla\eta(s))\,ds, \end{aligned} \tag{14}$$

$$\begin{aligned} &\int_0^T\langle c_s^n(s),\eta(s)\rangle\,ds + K_2\int_0^T(\nabla c^n(s),\nabla\eta(s))\,ds \\ &+K_2M\int_0^T(c^n(s)(1-c^n(s))\nabla\phi^n(s),\nabla\eta(s))\,ds \quad = \quad 0, \end{aligned} \tag{15}$$

for any $\eta \in L^2(0,T;H^1(\Omega))$.

Let us fix $t \in (0,T)$. Taking $\eta(s) = \phi^n(s)\chi_{(0,t)}(s)$ in (13), and observing that

$$\int_0^t\langle\phi_s^n(s),\phi^n(s)\rangle\,ds = \frac{1}{2}|\phi^n(t)|^2 - \frac{1}{2}|\phi_0^n|^2,$$

we have

$$\begin{aligned} &\frac{\alpha\varepsilon^2}{2}|\phi^n(t)|^2 + \varepsilon^2\int_0^t|\nabla\phi^n(s)|^2\,ds \\ = \;& \frac{\alpha\varepsilon^2}{2}|\phi_0^n|^2 + \frac{1}{2}\int_0^t|\phi^n(s)|^2\,ds - \frac{1}{2}\int_0^t\|\phi^n(s)\|_{L^4(\Omega)}^4\,ds \\ &+\frac{\beta}{C_V}\int_0^t(u^n(s),\phi^n(s))\,ds - \frac{\beta l}{2C_V}\int_0^t|\phi^n(s)|^2\,ds \\ &+\beta(\theta_B-\theta_A)\int_0^t(c^n(s),\phi^n(s))\,ds - \beta\theta_B\int_0^t(1,\phi^n(s))\,ds, \end{aligned}$$

and therefore, there exists a constant $C_1 > 0$, independent of n and T, such that

$$\begin{aligned} &\frac{\alpha\varepsilon^2}{2}|\phi^n(t)|^2 + \varepsilon^2 \int_0^t |\nabla\phi^n(s)|^2\, ds + \frac{1}{2}\int_0^t \|\phi^n(s)\|^4_{L^4(\Omega)}\, ds \\ \leq\ & \frac{\alpha\varepsilon^2}{2}|\phi_0^n|^2 + \frac{\beta}{2}\theta_B|\Omega|T \\ &+C_1 \int_0^t (|\phi^n(s)|^2 + |u^n(s)|^2 + |c^n(s)|^2)\, ds, \end{aligned} \tag{16}$$

for all $0 \leq t \leq T$.

Analogously, taking $\eta(s) = u^n(s)\chi_{(0,t)}(s)$ in (14), and observing that

$$\int_0^t \langle u_s^n(s), u^n(s)\rangle\, ds = \frac{1}{2}|u^n(t)|^2 - \frac{1}{2}|C_V\theta_0 + \frac{l}{2}\phi_0^n|^2,$$

we deduce

$$\begin{aligned} &\frac{1}{2}|u^n(t)|^2 + \frac{1}{C_V}\int_0^t (K_1(\phi^n(s))\nabla u^n(s), \nabla u^n(s))\, ds \\ =\ & \frac{1}{2}|C_V\theta_0 + \frac{l}{2}\phi_0^n|^2 + \frac{l}{2C_V}\int_0^t (K_1(\phi^n(s))\nabla\phi^n(s), \nabla u^n(s))\, ds, \end{aligned}$$

and therefore, taking into account (6) and using Young's inequality, one obtains

$$\begin{aligned} &|u^n(t)|^2 + \frac{\underline{k}_1}{C_V}\int_0^t |\nabla u^n(s)|^2\, ds \\ \leq\ & |C_V\theta_0 + \frac{l}{2}\phi_0^n|^2 + \frac{l^2\overline{k}_1^2}{4C_V\underline{k}_1}\int_0^t |\nabla\phi^n(s)|^2\, ds. \end{aligned} \tag{17}$$

Finally, taking $\eta(s) = c^n(s)\chi_{(0,t)}(s)$ in (15), and using that $0 \leq c^n(1-c^n) \leq 1$ a.e. in $\Omega \times (0,T)$, and

$$\int_0^t \langle c_s^n(s), c^n(s)\rangle\, ds = \frac{1}{2}|c^n(t)|^2 - \frac{1}{2}|c_0^n|^2,$$

we arrive at

$$|c^n(t)|^2 + K_2\int_0^t |\nabla c^n(s)|^2\, ds \leq |c_0^n|^2 + K_2M^2\int_0^t |\nabla\phi^n(s)|^2\, ds. \tag{18}$$

Now, adding (16), (17) multiplied by $\frac{\varepsilon^2 C_V\underline{k}_1}{l^2\overline{k}_1^2}$, and (18) multiplied by $\frac{\varepsilon^2}{4K_2M^2}$, and taking into account that the sequences $\{\phi_0^n\}_{n\geq 1}$ and $\{c_0^n\}_{n\geq 1}$ are bounded in $L^2(\Omega)$, we deduce that there exists a constant $C_2 > 0$, independent of n, t and T, such that

$$\begin{aligned} &|\phi^n(t)|^2 + |u^n(t)|^2 + |c^n(t)|^2 + \int_0^t \|\phi^n(s)\|^4_{L^4(\Omega)}\, ds \\ &+\int_0^t (|\nabla\phi^n(s)|^2 + |\nabla u^n(s)|^2 + |\nabla c^n(s)|^2)\, ds \\ \leq\ & C_2(1+T) + C_2\int_0^t (|\phi^n(s)|^2 + |u^n(s)|^2 + |c^n(s)|^2)\, ds \end{aligned}$$

for all $0 \leq t \leq T$, for any $n \geq 1$.

From this inequality and Gronwall lemma we infer that the sequence $\{\phi^n\}_n$ is bounded in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;L^4(\Omega))$, and the sequences

$\{u^n\}_n$ (and so $\{\theta^n\}_n$) and $\{c^n\}_n$ are bounded in $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$. It follows from (11) and (12) that the sequences $\{u_t^n\}_n$ and $\{c_t^n\}_n$ are bounded in $L^2(0,T;H^1(\Omega)')$. Since, evidently, $\{(\phi^n)^3\}_n$ is bounded in $L^{4/3}(0,T;L^{4/3}(\Omega))$, from (10) we obtain that $\{\phi_t^n\}_n$ is bounded in $L^2(0,T;H^1(\Omega)')+L^{4/3}(0,T;L^{4/3}(\Omega))$. Thus $\{\theta_t^n\}_n$ is also bounded in the same space.

Then, by taking into account that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, there exist three subsequences $\{\phi^\mu\}_\mu\subset\{\phi^n\}_n$, $\{\theta^\mu\}_\mu\subset\{\theta^n\}_n$, $\{c^\mu\}_\mu\subset\{c^n\}_n$, and five elements $\phi\in L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\cap L^4(0,T;L^4(\Omega))$, $\chi\in L^{4/3}(0,T;L^{4/3}(\Omega))$, and θ, c, and $u=C_V\theta+\dfrac{l}{2}\phi$, belonging to $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ such that

$$\begin{cases} \phi^\mu\rightharpoonup\phi \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)) \quad \text{and in } \ L^4(0,T;L^4(\Omega)),\\ \phi^\mu\overset{*}{\rightharpoonup}\phi \quad \text{weakly star in } \ L^\infty(0,T;L^2(\Omega)),\\ \phi^\mu\to\phi \quad \text{strongly in } \ L^2(0,T;L^2(\Omega)),\\ \phi^\mu\to\phi \quad \text{a.e.} \quad \text{in } \ \Omega\times(0,T),\\ \phi_t^\mu\rightharpoonup\phi_t \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)')+L^{4/3}(0,T;L^{4/3}(\Omega)),\\ (\phi^\mu)^3\rightharpoonup\chi \quad \text{weakly in } \ L^{4/3}(0,T;L^{4/3}(\Omega)),\end{cases} \tag{19}$$

$$\begin{cases} u^\mu\rightharpoonup u \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)),\\ u^\mu\overset{*}{\rightharpoonup}u \quad \text{weakly star in } \ L^\infty(0,T;L^2(\Omega)),\\ u^\mu\to u \quad \text{strongly in } \ L^2(0,T;L^2(\Omega)),\\ u^\mu\to u \quad \text{a.e.} \quad \text{in } \ \Omega\times(0,T),\\ u_t^\mu\rightharpoonup u_t \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)'),\end{cases} \tag{20}$$

$$\begin{cases} \theta^\mu\rightharpoonup\theta \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)),\\ \theta^\mu\overset{*}{\rightharpoonup}\theta \quad \text{weakly star in } \ L^\infty(0,T;L^2(\Omega)),\\ \theta^\mu\to\theta \quad \text{strongly in } \ L^2(0,T;L^2(\Omega)),\\ \theta^\mu\to\theta \quad \text{a.e.} \quad \text{in } \ \Omega\times(0,T),\\ \theta_t^\mu\rightharpoonup\theta_t \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)')+L^{4/3}(0,T;L^{4/3}(\Omega)),\end{cases} \tag{21}$$

$$\begin{cases} c^\mu\rightharpoonup c \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)),\\ c^\mu\overset{*}{\rightharpoonup}c \quad \text{weakly star in } \ L^\infty(0,T;L^2(\Omega)),\\ c^\mu\to c \quad \text{strongly in } \ L^2(0,T;L^2(\Omega)),\\ c^\mu\to c \quad \text{a.e.} \quad \text{in } \ \Omega\times(0,T),\\ c_t^\mu\rightharpoonup c_t \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)').\end{cases} \tag{22}$$

From (19) and Lemma 1.3, page 12 in [14], one obtains that $\chi=\phi^3$. Therefore, passing to the limit in (10) we find

$$\alpha\varepsilon^2\phi_t-\varepsilon^2\Delta\phi=\frac{1}{2}(\phi-\phi^3)+\beta(\theta+(\theta_B-\theta_A)c-\theta_B),$$

in the sense of $\mathcal{D}'(\Omega\times(0,T))$. As $\phi\in L^2(0,T;H^1(\Omega))\cap L^4(0,T;L^4(\Omega))$ and $\phi_t\in L^2(0,T;H^1(\Omega)')+L^{4/3}(0,T;L^{4/3}(\Omega))$, then $\phi\in C([0,T];L^2(\Omega))$. In fact, ϕ also satisfies (7), and $\dfrac{\partial\phi}{\partial\nu}=0$ on $\partial\Omega\times(0,T)$ in a generalized sense.

The initial condition $\phi(0)=\phi_0$ is an easy consequence of the equality

$$\phi^\mu(t)=\phi_0^\mu+\int_0^t\phi_s^\mu(s)\,ds\quad\forall\,t\geq 0,$$

in the sense $H^1(\Omega)'+L^{4/3}(\Omega)$, (19), and the fact that $\phi_0^\mu\to\phi_0$ in $L^2(\Omega)$.

On the other hand, from (19) and the fact that the function K_1 is globally Lipschitz continuous, we have that $K_1(\phi^\mu)\to K_1(\phi)$ strongly in $L^2(0,T;L^2(\Omega))$. Thus, it follows from (21) that

$$K_1(\phi^\mu)\nabla\theta^\mu\rightharpoonup K_1(\phi)\nabla\theta\quad\text{weakly in } L^1(0,T;L^1(\Omega)).$$

Note that, from (6), we have

$$|K_1(\phi^\mu)\nabla\theta^\mu|\leq\overline{k}_1|\nabla\theta^\mu|,$$

and therefore $\{K_1(\phi^\mu)\nabla\theta^\mu\}_\mu$ is bounded in $L^2(0,T;L^2(\Omega))$. Consequently,

$$K_1(\phi^\mu)\nabla\theta^\mu\rightharpoonup K_1(\phi)\nabla\theta\quad\text{weakly in } L^2(0,T;L^2(\Omega)).\tag{23}$$

Now, we can deduce from (11), (19), (21) and (23), that (8) holds. Since $u\in L^2(0,T;H^1(\Omega))$ and $u_t\in L^2(0,T;H^1(\Omega)')$, we have that $u\in C([0,T];L^2(\Omega))$. As $\phi\in C([0,T];L^2(\Omega))$, we infer that $\theta=\frac{1}{C_V}(u-\frac{l}{2}\phi)$ belongs to the same space. Also, the equality $\theta(0)=\theta_0$ can be deduced analogously to the case of ϕ.

Next, for the function c, observe first that since $c^\mu\to c$ and $0\leq c^\mu\leq 1$ a.e. in $\Omega\times(0,T)$, we have that

$$0\leq c\leq 1\quad\text{a.e. in }\Omega\times(0,T).$$

The Lebesgue dominated convergence Theorem ensures that

$$c^\mu(1-c^\mu)\to c(1-c)\quad\text{strongly in } L^2(0,T;L^2(\Omega)).$$

Thus, by using (22), we deduce that

$$c^\mu(1-c^\mu)\nabla\phi^\mu\rightharpoonup c(1-c)\nabla\phi\quad\text{weakly in } L^1(0,T;L^1(\Omega)).$$

Observe that,

$$|c^\mu(1-c^\mu)\nabla\phi^\mu|\leq|\nabla\phi^\mu|,$$

so that $\{c^\mu(1-c^\mu)\nabla\phi^\mu\}_\mu$ is bounded in $L^2(0,T;L^2(\Omega))$. Consequently,

$$c^\mu(1-c^\mu)\nabla\phi^\mu\rightharpoonup c(1-c)\nabla\phi\quad\text{weakly in } L^2(0,T;L^2(\Omega)).\tag{24}$$

Convergences (19), (22), and (24) allow us to pass to the limit in (12) obtaining (9). By reasoning as for u, we infer that $c\in C([0,T];L^2(\Omega))$ and $c(0)=c_0$.

The above result can be carried out in any interval (remember that T was fixed but arbitrary). By the continuity of the functions ϕ, θ, and c, and since the problem (1)–(5) is autonomous, one can concatenate solutions in intervals $[0,T]$, $[T,2T]$, etc, obtaining by induction solutions defined over all Q.

Finally, if $\phi_0\in H^1(\Omega)$, taking into account well known regularity results (just using the special basis and a posteriori regularity in the Galerkin scheme; for instance, cf. [14, 15]), we obtain that $\phi \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega))$, $\phi_t \in L^2(0,T;L^2(\Omega))$, ϕ satisfies (1) a.e. in Q, and $\dfrac{\partial\phi}{\partial\nu} = 0$ a.e. on Σ. □

The regularity result at the end of the above theorem for more regular data, and the fact that any solution (ϕ,θ,c), even with data in $(L^2(\Omega))^2 \times L^2(\Omega;[0,1])$, satisfies $\phi \in L^2(0,T;H^1(\Omega))$ for any $T>0$, points out a regularizing effect in the problem. Actually, we have the following result.

Proposition 2. *Assume that (6) holds. Then, any solution (ϕ,θ,c) of (1)–(5) with initial data $(\phi_0,\theta_0,c_0) \in (L^2(\Omega))^2 \times L^2(\Omega;[0,1])$ satisfies*

$$\phi \in C((0,+\infty);H^1(\Omega)) \cap L^2(\epsilon,T;H^2(\Omega)) \ \forall \epsilon, T>0, \qquad \frac{\partial\phi}{\partial\nu} = 0 \ a.e. \ on\, \Sigma,$$

and moreover, (ϕ,θ,c) satisfies (1) a.e. in Q.

Proof. Consider any solution (ϕ,θ,c) to (1)–(5) with data $(\phi_0,\theta_0,c_0) \in (L^2(\Omega))^2 \times L^2(\Omega;[0,1])$. Fix any positive value $\epsilon_1 > 0$. By Theorem 1 we have that $\phi \in L^2(0,\epsilon_1;H^1(\Omega))$, so a.e. in $(0,\epsilon_1)$, $\phi(t) \in H^1(\Omega)$. Consider one of these values, $\epsilon_2 \in (0,\epsilon_1)$, such that $\phi(\epsilon_2) \in H^1(\Omega)$.

Observe that if we define $\tilde\theta(t) = \theta(\epsilon_2+t)$ and $\tilde c(t) = c(\epsilon_2+t)$, then $\tilde\phi(t) = \phi(\epsilon_2+t)$ is the unique solution to the problem

$$\alpha\varepsilon^2\tilde\phi_t - \varepsilon^2\Delta\tilde\phi = \frac{1}{2}(\tilde\phi - \tilde\phi^3) + \beta(\tilde\theta - \tilde c\theta_A - (1-\tilde c)\theta_B) \quad \text{in} \quad Q,$$

with $\dfrac{\partial\tilde\phi}{\partial\nu} = 0$ on Σ and $\tilde\phi(0) = \phi(\epsilon_2)$.

The regularity of the solution of this problem is well known (see above): $\tilde\phi \in L^2(0,T;H^2(\Omega)) \cap C([0,+\infty);H^1(\Omega))$ for any $T>0$ and $\dfrac{\partial\tilde\phi}{\partial\nu} = 0$ *a.e.* on Σ.

Therefore, $\phi \in C([\epsilon_2,+\infty);H^1(\Omega)) \cap L^2(\epsilon_2,T;H^2(\Omega))$ for any $T>0$ and normal derivative null a.e. Now, the proof finishes repeating the argument with ϵ_1 substituted by ϵ_1^n with $\{\epsilon_1^n\}_n$ a sequence of strictly decreasing positive values with $\lim\limits_{n\to+\infty} \epsilon_1^n = 0$. □

We present another interesting result, which provides an invariant all through the time for each solution, that will be important for the study of the asymptotic behaviour of our problem.

Proposition 3. *Assume that (6) holds. Consider any solution (ϕ,θ,c) of (1)–(5) with initial data in $(L^2(\Omega))^2 \times L^2(\Omega;[0,1])$. Then, the function*

$$\mathbb{R}_+ \ni t \mapsto \int_\Omega u(t,x)dx,$$

where $u = C_V\theta + \dfrac{l}{2}\phi$, is constant.

Proof. Fix any $T>0$. We have to check that the derivative

$$\frac{d}{dt}(u(t),1) = 0 \quad \text{in } \mathcal{D}'(0,T).$$

By the integration by parts formula (e.g. see [6, Vol.3]) we have the equality

$$\frac{d}{dt}(u(t),1) = \langle u'(t),1\rangle,$$

and this is zero by (8). □

Remark 4. *From the above result, any solution* (ϕ, θ, c) *with initial data* $(\phi_0, \theta_0, c_0) \in (L^2(\Omega))^2 \times L^2(\Omega; [0,1])$ *satisfies*

$$\frac{1}{|\Omega|}\left(\int_\Omega u(t,x)dx\right)^2 = \frac{1}{|\Omega|}\left(\int_\Omega u_0(x)dx\right)^2, \quad \forall t \in \mathbb{R}_+,$$

where we have denoted obviously $u_0 = C_V\theta_0 + \frac{l}{2}\phi_0$. *This quantity will be useful in the* $H^1(\Omega)$*-framework to relate the* $L^2(\Omega)$*-norm of a function with the norm of its gradient.*

More exactly, we recall that the Poincaré-Wirtinger inequality (e.g. cf. [9]*) says that there exists a constant* $C_\Omega > 0$ *such that for any element* $\chi \in H^1(\Omega)$ *it holds*

$$|\chi|^2 - \frac{1}{|\Omega|}\left(\int_\Omega \chi(x)dx\right)^2 = \left|\chi - \frac{1}{|\Omega|}\int_\Omega \chi(x)dx\right|^2 \le C_\Omega|\nabla\chi|^2.$$

3. Abstract multi-valued dynamical systems

In this section we recall briefly some basic statements from multi-valued dynamical systems and their asymptotic behaviour (cf. [16] and references therein). This will be important in order to state our problem in a suitable dynamical system framework since uniqueness of solution for (1)–(5) is unknown. Then, we establish the essential properties involved to ensure the existence of attractors.

Definition 5. *Given a metric space* $(\mathcal{X}, d)$, *a multi-valued map* $\mathcal{G} : \mathbb{R}_+ \times \mathcal{X} \to P(\mathcal{X})$ *is called a multi-valued semiflow, and will be denoted* $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\ge0})$, *if*

a) $\mathcal{G}(0,\cdot) = \mathrm{Id}$ *(identity map),*

b) *for any pair* t_1, $t_2 \ge 0$ *and for all* $x \in \mathcal{X}$,

$$\mathcal{G}(t_1 + t_2, x) \subset \mathcal{G}(t_1, \mathcal{G}(t_2, x)), \quad \textit{where} \quad \mathcal{G}(t, A) = \bigcup_{a\in A} \mathcal{G}(t, a).$$

When the above inclusion is an equality, it is said that the multi-valued semiflow is strict.

Let us observe that the continuity notion for multi-valued maps is not unique, and the upper semicontinuity is the suitable notion for results on attractors (see below). A multi-valued map $F : \mathcal{X} \to P(\mathcal{X})$ is upper semicontinuous if for every $x \in \mathcal{X}$ and every neighbourhood M of $F(x)$, there exists a neighbourhood N of x such that $F(y) \subset M$ for any $y \in N$. Note that when the semiflow is single valued, we recover the usual notion of continuity.

Definition 6. *A multi-valued semiflow* $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\ge0})$ *is called pointwise dissipative if there exists* $B \in B(\mathcal{X})$ *that attracts the dynamics starting at all single points, i.e.*

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{X}}(\mathcal{G}(t, x), B) = 0 \quad \forall x \in \mathcal{X}.$$

It is called asymptotically compact if for any $B \in B(\mathcal{X})$ *and any sequence* $\{t_n\}_n$ *with* $t_n \to +\infty$, *any sequence* $\{\psi_n\}_n$ *with* $\psi_n \in \mathcal{G}(t_n, B)$ *possesses a converging subsequence in* $\mathcal{X}$.

The following result was stated in [16] for complete metric spaces, but it really does not need the completeness. It also contains the definition of the well known concept of global attractor.

Theorem 7. [cf. [16, Th.3 & Remark 8]] *Let $(\mathcal{X}, d)$ be a metric space, and $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$ be a pointwise dissipative and asymptotically compact strict multi-valued semiflow. Suppose that $\mathcal{G}(t,\cdot) : \mathcal{X} \to C(\mathcal{X})$ is upper semicontinuous for any $t \geq 0$. Then $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$ possesses the global attractor $\mathcal{A}$, that is, a compact invariant set, $\mathcal{G}(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, that attracts all bounded sets:*

$$\lim_{t\to+\infty} \operatorname{dist}_{\mathcal{X}}(\mathcal{G}(t,B), \mathcal{A}) = 0 \qquad \forall B \in B(\mathcal{X}).$$

It is minimal among all closed sets attracting each bounded set.

There exists a more restrictive way to obtain a global attractor than the above result. We introduce it since these sufficient conditions will hold in our situation.

Definition 8. *A set $B_0 \subset \mathcal{X}$ is said to be an absorbing set for the multi-valued semiflow $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$ if for any $B \in B(\mathcal{X})$, there exists a time $T(B)$ such that $\mathcal{G}(t,B) \subset B_0, \quad \forall\, t \geq T(B)$.*

We say that $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$ is compact if for any $T > 0$, and any $B \in B(\mathcal{X})$, the set $\mathcal{G}(T,B)$ is relatively compact in $\mathcal{X}$.

Remark 9. *If $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$ is a strict multi-valued semiflow, compact, with $\mathcal{G}(t,\cdot) : \mathcal{X} \to C(\mathcal{X})$ upper semicontinuous for any $t \geq 0$, and there exists a bounded absorbing set, then assumptions (and thesis) in* Theorem 7 *hold.*

4. Semiflows for phase-field model and the absorbing property

Theorem 1 allows us to define a multi-valued map using the set of solutions for (1)–(5) corresponding to a triplet of initial data. The multi-valued performance is due to the fact that uniqueness of solution for the problem is unknown.

Namely, denote $D(\phi_0, \theta_0, c_0)$ the set of global solutions to (1)–(5) with initial conditions $(\phi_0, \theta_0, c_0) \in (L^2(\Omega))^2 \times L^2(\Omega; [0,1])$. Now, we define

$$G(t, \phi_0, \theta_0, c_0) = \{(\phi(t), \theta(t), c(t)) \ : \ (\phi, \theta, c) \in D(\phi_0, \theta_0, c_0)\},$$

which is well defined by the continuity in time of solutions. Indeed, Theorem 1 combined with Proposition 3 allows to construct several multi-valued semiflows, always with the same map, but from different suitable metric spaces into themselves.

Definition 10. *Denote*

$$L^2_\gamma = \{(\phi, \theta, c) \in (L^2(\Omega))^2 \times L^2(\Omega; [0,1]) \ : \ \frac{1}{|\Omega|}\int_\Omega \left(C_V\theta + \frac{l}{2}\phi\right) dx = \gamma\}, \quad \forall \gamma \in \mathbb{R},$$

and

$$\mathcal{L}^2_\rho = \bigcup_{|\gamma|\leq\rho} L^2_\gamma \ = \ \Big\{(\phi, \theta, c) \in (L^2(\Omega))^2 \times L^2(\Omega; [0,1]) \ : \ \frac{1}{|\Omega|}\left|\int_\Omega \left(C_V\theta + \frac{l}{2}\phi\right) dx\right| \leq \rho\Big\}, \quad \forall \rho \in \mathbb{R}_+,$$

which are complete metric spaces with the distance induced by the $(L^2(\Omega))^3$-norm.

Denote also

$$H^1_\gamma = (H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega; [0,1])) \cap L^2_\gamma, \quad \forall \gamma \in \mathbb{R},$$

and

$$\mathcal{H}^1_\rho = (H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega; [0,1])) \cap \mathcal{L}^2_\rho, \quad \forall \rho \in \mathbb{R}_+,$$

which are also complete metric spaces with the distance induced by the $H^1(\Omega) \times (L^2(\Omega))^2$-norm.

From Theorem 1 and Proposition 3 is not difficult to conclude that

Proposition 11. *Assume that (6) holds. Then, the following pairs, formed by the multi-valued map G and different metric spaces, define strict multi-valued semi-flows:*

$$\begin{gathered}\left((L^2(\Omega))^2 \times L^2(\Omega;[0,1]), \{G(t)\}_{t\geq 0}\right), \\ \left(H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1]), \{G(t)\}_{t\geq 0}\right), \\ \left(L^2_\gamma, \{G(t)\}_{t\geq 0}\right), \quad \text{and} \quad \left(H^1_\gamma, \{G(t)\}_{t\geq 0}\right), \quad \forall \gamma \in \mathbb{R}, \\ \left(\mathcal{L}^2_\rho, \{G(t)\}_{t\geq 0}\right), \quad \text{and} \quad \left(\mathcal{H}^1_\rho, \{G(t)\}_{t\geq 0}\right), \quad \forall \rho \in \mathbb{R}_+.\end{gathered}$$

Remark 12. *The multi-valued semiflows stated in the spaces L^2_γ and H^1_γ, although mathematically correct, do not seem to represent a realistic situation. One would aim that a small perturbation of an initial point contains a ball, which is not possible in these spaces. That suggests the introduction of $\mathcal{L}^2_\rho$ and $\mathcal{H}^1_\rho$, where this works well.*

Observe that Proposition 3 gives sense to both possibilities. Therefore, we will carry on all of them, but concentrating mainly in $\mathcal{L}^2_\rho$ and $\mathcal{H}^1_\rho$; the proved properties for them will be automatically inherited by L^2_γ and H^1_γ. At the end of the paper we give a complete answer to the relationship between all these dynamics.

In order to find out if some kind of absorbing property holds for any of the above multi-valued semiflows, we obtain estimates for the solutions in (essentially) the two possible situations, i.e. with non-regular ($(L^2)^3$) and regular ($H^1 \times (L^2)^2$) data.

Proposition 13. *Assume that (6) holds. Concerning the solutions of problem (1)–(5), the following estimates hold:*

(a) *There exists a positive constant C_5 such that for any solution (ϕ, θ, c) with initial data $(\phi_0, \theta_0, c_0) \in (L^2(\Omega))^2 \times L^2(\Omega;[0,1])$, there exists a positive value C_4, depending on $u_0 = C_V\theta_0 + \frac{l}{2}\phi_0$, such that*

$$\begin{aligned}&\frac{\alpha\varepsilon^2}{2}|\phi(t)|^2 + \frac{2\varepsilon^2 C_V \underline{k}_1}{\overline{k}_1^2 l^2}|C_V\theta(t) + \frac{l}{2}\phi(t)|^2 \\ \leq\ &\left(\frac{\alpha\varepsilon^2}{2}|\phi_0|^2 + \frac{2\varepsilon^2 C_V \underline{k}_1}{\overline{k}_1^2 l^2}|u_0|^2\right) e^{-C_5 t} + \frac{C_4(u_0)}{C_5}.\end{aligned} \tag{25}$$

More exactly, the value C_4 depends on the average in Ω of the function u_0.

(b) *There also exist positive constants C_6, C_7, and C_8, such that if $(\phi_0, \theta_0, c_0) \in H^1(\Omega) \times L^2(\Omega)^2 \times L^2(\Omega;[0,1])$, any associated solution (ϕ, θ, c) satisfies*

$$\begin{aligned}&\frac{\alpha\varepsilon^2}{2}|\phi(t)|^2 + \frac{2\varepsilon^2 C_V \overline{k}_1}{\underline{k}_1^2 l^2}|C_V\theta(t) + \frac{l}{2}\phi(t)|^2 + \frac{\alpha\varepsilon^2 C_6}{2}|\nabla\phi(t)|^2 \\ \leq\ &\left(\frac{\alpha\varepsilon^2}{2}|\phi_0|^2 + \frac{2\varepsilon^2 C_V \overline{k}_1}{\underline{k}_1^2 l^2}|u_0|^2 + \frac{\alpha\varepsilon^2 C_6}{2}|\nabla\phi_0|^2\right) e^{-C_7 t} + \frac{C_4(u_0) + C_8}{C_7}.\end{aligned} \tag{26}$$

Remark 14. (i) *Besides the decreasing exponential, the additional term in the right hand side of (25) and (26), forced by the Neumann boundary condition and the necessity of relating the L^2-norm of a function and its gradient (see below, and also Proposition 3 and Remark 4), it is not clear that the multi-valued semiflows $((L^2(\Omega))^2 \times L^2(\Omega;[0,1]), \{G(t)\}_{t\geq 0})$ and $(H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1]), \{G(t)\}_{t\geq 0})$,* *have absorbing sets or are pointwise dissipative. This suggests the level-set formulation using the spaces introduced in Definition 10.*

(ii) *We point out that any solution satisfies that $c \in [0,1]$ a.e., so $c \in L^\infty(Q)$, and there is no need of additional estimates for boundedness on this third variable.*

Proof. [of Proposition 13] As already commented (cf. Remark 4), it will be useful in obtaining estimates to recover the change of variables that we have used in Theorem 1.

So, consider the variable $u = C_V\theta + \frac{l}{2}\phi$. We could rewrite the problem (1)–(5) in terms of the variables (ϕ, u, c) (indeed it was implicitly done in (13)–(15)), but for brevity we do not write it down here.

Step 1: L^2-estimates. We prove the claim (a). For the sake of brevity in the equations below we will use the derivative instead of the integral form.

Taking ϕ as test function in (7) and applying the Young inequality with arbitrary constants to fix later, we obtain

$$\begin{aligned} &\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\phi|^2 + \varepsilon^2|\nabla\phi|^2 + \frac{1}{2}\int_\Omega(\phi^4 - \phi^2)dx + \frac{\beta l}{2C_V}|\phi|^2 \\ = \;& \int_\Omega\left(\frac{\beta}{C_V}u\phi + \beta(\theta_B - \theta_A)c\phi - \beta\theta_B\phi\right)dx \\ \le \;& \epsilon|u|^2 + \delta\|\phi\|^4_{L^4(\Omega)} + \frac{\beta^4}{64\delta\epsilon^2 C_V^4} + \epsilon'|\phi|^2 + \frac{1}{4\epsilon'}|\beta(\theta_B - \theta_A)c - \beta\theta_B|^2 \\ \le \;& \epsilon|u|^2 + \delta\|\phi\|^4_{L^4(\Omega)} + \epsilon'|\phi|^2 + C_3, \end{aligned}$$

where

$$C_3 = \frac{\beta^4}{64\delta\epsilon^2 C_V^4} + \frac{\beta^2|\Omega|(\theta_B^2 + \theta_A^2)}{2\epsilon'}.$$

Choosing $\delta = \epsilon' = 1/4$ we deduce

$$\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\phi|^2 + \varepsilon^2|\nabla\phi|^2 + \frac{1}{4}\int_\Omega(\phi^4 - 3\phi^2)dx + \frac{\beta l}{2C_V}|\phi|^2 \le \epsilon|u|^2 + C_3. \tag{27}$$

Now, taking u as test function in (8) we deduce

$$\frac{C_V}{2}\frac{d}{dt}|u|^2 + \int_\Omega K_1(\phi)|\nabla u|^2 dx = \frac{l}{2}\int_\Omega K_1(\phi)\nabla\phi\cdot\nabla u dx,$$

and using assumption (6) of boundedness for K_1, we obtain

$$\frac{C_V}{2}\frac{d}{dt}|u|^2 + \underline{k}_1|\nabla u|^2 \le \frac{l^2\overline{k}_1^2}{8\underline{k}_1}|\nabla\phi|^2 + \frac{\underline{k}_1}{2}|\nabla u|^2.$$

So, arranging terms and multiplying by $\frac{2\underline{k}_1\varepsilon^2}{\overline{k}_1^2 l^2}$ we conclude

$$\frac{2\underline{k}_1\varepsilon^2 C_V}{\overline{k}_1^2 l^2}\frac{d}{dt}|u|^2 + \frac{2\underline{k}_1^2\varepsilon^2}{\overline{k}_1^2 l^2}|\nabla u|^2 \le \frac{\varepsilon^2}{2}|\nabla\phi|^2, \tag{28}$$

which added to (27) gives

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2 + \frac{2\varepsilon^2 C_V\underline{k}_1}{l^2\overline{k}_1^2}|u|^2\right) + \frac{\varepsilon^2}{2}|\nabla\phi|^2 \\ &+\frac{1}{4}\int_\Omega(\phi^4 - 3\phi^2)dx + \frac{\beta l}{2C_V}|\phi|^2 + \frac{2\underline{k}_1^2\varepsilon^2}{\overline{k}_1^2 l^2}|\nabla u|^2 \le \epsilon|u|^2 + C_3. \end{aligned} \tag{29}$$

Now, to compare the quantities $|u|$ and $|\nabla u|$, we use the Poincaré-Wirtinger inequality and the invariant quantity $\int_\Omega u(t,x)dx \equiv \int_\Omega u_0(x)dx$ for all $t \geq 0$, established in Proposition 3 (see also Remark 4).

So, and taking $\epsilon = \dfrac{\underline{k}_1^2\varepsilon^2}{\overline{k}_1^2 l^2 C_\Omega}$ in (29) combined with the inequality $x^4 - 3x^2 \geq x^2 - 4$ for all $x \in \mathbb{R}$, we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2 + \frac{2\varepsilon^2 C_V \underline{k}_1}{l^2\overline{k}_1^2}|u|^2\right) + \frac{\varepsilon^2}{2}|\nabla\phi|^2 \\
&+\left(\frac{1}{4} + \frac{\beta l}{2C_V}\right)|\phi|^2 + \frac{\underline{k}_1^2\varepsilon^2}{\overline{k}_1^2 l^2 C_\Omega}|u|^2 \leq C_4(u_0),
\end{aligned} \tag{30}$$

where

$$C_4(u_0) = |\Omega| + C_3 + \frac{2\varepsilon^2\underline{k}_1^2}{\overline{k}_1^2 l^2|\Omega| C_\Omega}\left(\int_\Omega u_0 dx\right)^2.$$

Now, an inequality of the type

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2 + \frac{2C_V\underline{k}_1\varepsilon^2}{\overline{k}_1^2 l^2}|u|^2\right) \\
&+C_5\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2 + \frac{2C_V\underline{k}_1\varepsilon^2}{\overline{k}_1^2 l^2}|u|^2\right) \leq C_4(u_0),
\end{aligned}$$

is easy to conclude from (30) choosing

$$0 < C_5 < \min\left\{\frac{1}{\alpha\varepsilon^2}\left(\frac{1}{2} + \frac{\beta l}{C_V}\right), \frac{\underline{k}_1}{2C_V C_\Omega}\right\}.$$

whence (25) follows.

Step 2: H^1-estimate for ϕ. We will obtain an extra estimate for $\nabla\phi$ that complements the obtained in Step 1 to conclude (b). We make the most of the extra regularity that we have for solutions with regular data (cf. Theorem 1).

Multiplying equation (1) by $-\Delta\phi$, we obtain

$$\begin{aligned}
&\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\nabla\phi|^2 + \varepsilon^2|\Delta\phi|^2 + \frac{3}{2}\int_\Omega \phi^2|\nabla\phi|^2 dx + \frac{1}{2}\int_\Omega \phi\Delta\phi dx + \frac{\beta l}{2C_V}|\nabla\phi|^2 \\
= \quad & -\frac{\beta}{C_V}\int_\Omega u\Delta\phi dx - \beta(\theta_B - \theta_A)\int_\Omega c\Delta\phi dx,
\end{aligned}$$

since the term $-\beta\theta_B\int_\Omega \Delta\phi dx$ disappears integrating by parts.

Applying again the Young inequality and the fact that $c \in [0,1]$ in the right hand side, we deduce

$$\begin{aligned}
&\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\nabla\phi|^2 + \frac{\varepsilon^2}{2}|\Delta\phi|^2 + \frac{\beta l}{2C_V}|\nabla\phi|^2 \\
\leq \quad & \frac{\beta^2}{\varepsilon^2 C_V^2}|u|^2 + \frac{\beta^2(\theta_B - \theta_A)^2|\Omega|}{\varepsilon^2} - \frac{1}{2}\int_\Omega \phi\Delta\phi dx \\
\leq \quad & \frac{\beta^2}{\varepsilon^2 C_V^2}|u|^2 + \frac{\beta^2(\theta_B - \theta_A)^2|\Omega|}{\varepsilon^2} + \frac{\varepsilon^2}{4}|\Delta\phi|^2 + \frac{1}{4\varepsilon^2}|\phi|^2.
\end{aligned}$$

So, in particular, neglecting one term in the left hand side, we obtain

$$\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\nabla\phi|^2+\frac{\beta l}{2C_V}|\nabla\phi|^2\le\frac{\beta^2}{\varepsilon^2C_V^2}|u|^2+\frac{\beta^2(\theta_B-\theta_A)^2|\Omega|}{\varepsilon^2}+\frac{1}{4\varepsilon^2}|\phi|^2. \tag{31}$$

Multiplying this inequality by a suitable constant C_6 to be fixed later on, and adding to (30) it yields

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2+\frac{2\varepsilon^2C_V\underline{k}_1}{\underline{k}_1^2l^2}|u|^2+\frac{\alpha\varepsilon^2C_6}{2}|\nabla\phi|^2\right)\\&+\left(\frac{1}{4}+\frac{\beta l}{2C_V}-\frac{C_6}{4\varepsilon^2}\right)|\phi|^2+\left(\frac{\underline{k}_1^2\varepsilon^2}{\overline{k}_1^2l^2C_\Omega}-\frac{\beta^2C_6}{\varepsilon^2C_V^2}\right)|u|^2\\&+\left(\frac{\varepsilon^2}{2}+\frac{\beta lC_6}{2C_V}\right)|\nabla\phi|^2\le C_4(u_0)+\frac{\beta^2}{\varepsilon^2}(\theta_B-\theta_A)^2C_6|\Omega|.\end{aligned}$$

Again we aim to obtain from here an inequality of the type

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2+\frac{2\varepsilon^2C_V\underline{k}_1}{\underline{k}_1^2l^2}|u|^2+\frac{\alpha\varepsilon^2C_6}{2}|\nabla\phi|^2\right)\\&+C_7\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2+\frac{2\varepsilon^2C_V\underline{k}_1}{\underline{k}_1^2l^2}|u|^2+\frac{\alpha\varepsilon^2C_6}{2}|\nabla\phi|^2\right)\\\le\;&C_4(u_0)+\frac{\beta^2}{\varepsilon^2}(\theta_B-\theta_A)^2C_6|\Omega|,\end{aligned}\tag{32}$$

with $C_7>0$, which is possible comparing coefficients and taking

$$0<C_6<\min\left\{\varepsilon^2\left(1+\frac{2\beta l}{C_V}\right),\frac{\underline{k}_1^2\varepsilon^4C_V^2}{\beta^2\overline{k}_1^2l^2C_\Omega}\right\},$$

and then

$$C_7=\min\left\{\frac{2}{\alpha\varepsilon^2}\left(\frac{1}{4}+\frac{\beta l}{2C_V}-\frac{C_6}{4\varepsilon^2}\right),\frac{\overline{k}_1^2l^2}{2\varepsilon^2C_V\underline{k}_1}\left(\frac{\underline{k}_1^2\varepsilon^2}{\overline{k}_1^2l^2C_\Omega}-\frac{\beta^2C_6}{\varepsilon^2C_V^2}\right),\frac{\beta l}{\alpha\varepsilon^2C_V}\right\}.$$

Now, from (32) it is easy to conclude (26), denoting $C_8=\frac{\beta^2}{\varepsilon^2}(\theta_B-\theta_A)^2C_6|\Omega|$. □

As an immediate consequence of Proposition 13, we have the following result.

Corollary 15. *Under assumption (6), the multi-valued semiflows* $\left(L^2_\gamma,\{G(t)\}_{t\ge0}\right)$ *and* $\left(H^1_\gamma,\{G(t)\}_{t\ge0}\right)$ *, for all* $\gamma\in\mathbb{R}$*, and* $\left(\mathcal{L}^2_\rho,\{G(t)\}_{t\ge0}\right)$ *, and* $\left(\mathcal{H}^1_\rho,\{G(t)\}_{t\ge0}\right)$ *, for all* $\rho\in\mathbb{R}_+$ *have bounded absorbing sets in their respective phase-spaces.*

To conclude this section, we give another result that will be useful for the analysis of the compact properties of the semiflows, and also for the study of attractors.

Proposition 16. *Assume that (6) holds and consider any value* $T>0$ *and any bounded set* B *from* $(L^2(\Omega))^2\times L^2(\Omega;[0,1])$*. Then,* $G(T,B)$ *is bounded in* $H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1])$*.*

Proof. By (25) and the fact that c takes values in $[0,1]$, we only must care about the $L^2(\Omega)$-norm of $\nabla\phi$ for any solution (ϕ,θ,c) with initial values in B.

So, fix one initial data $(\phi_0,\theta_0,c_0)\in B$ and consider any solution $(\phi,\theta,c)\in D(\phi_0,\theta_0,c_0)$. For any positive time $0<T'<T$ one has by Proposition 2 that

$\phi(T') \in H^1(\Omega)$. So, by the regularizing effect of the problem, with initial data $(\phi(T'), \theta(T'), c(T'))$, it makes sense to multiply (1) by $-\Delta\phi$, obtaining

$$\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}\left(|\nabla\phi|^2\right) + \varepsilon^2|\Delta\phi|^2 - \frac{1}{2}(\phi^3, \Delta\phi) = -\frac{1}{2}(\phi, \Delta\phi) - (h, \Delta\phi),$$

where we have denoted for brevity $h(\cdot) = \beta(\theta(T'+\cdot) - c(T'+\cdot)\theta_A - (1-c(T'+\cdot))\theta_B)$. Since integrating by parts $-(\phi^3, \Delta\phi) = 3(\phi^2, |\nabla\phi|^2)$, and this is a positive term in the left hand side, we can neglect it. Hence, integrating by parts and using the Young inequality in the right hand side, we deduce that

$$\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}\left(|\nabla\phi|^2\right) + \varepsilon^2|\Delta\phi|^2 \leq \frac{1}{2}|\nabla\phi|^2 + \varepsilon^2|\Delta\phi|^2 + \frac{1}{4\varepsilon^2}|h|^2,$$

whence

$$\frac{d}{dt}\left(|\nabla\phi|^2\right) \leq \frac{1}{\alpha\varepsilon^2}|\nabla\phi|^2 + \frac{1}{2\varepsilon^4\alpha}|h|^2.$$

By integrating in time we find

$$|\nabla\phi(T)|^2 \leq |\nabla\phi(t)|^2 + \frac{1}{\alpha\varepsilon^2}\int_t^T |\nabla\phi(s)|^2 ds + \frac{T-t}{2\varepsilon^4\alpha}\|h\|^2_{L^\infty(0,T;L^2(\Omega))}, \ \forall t \in [T', T].$$

Integrating again now in the variable t on $[T', T]$ one obtains

$$(T-T')|\nabla\phi(T)|^2 \leq \left(1 + \frac{T-T'}{\alpha\varepsilon^2}\right)\int_{T'}^T |\nabla\phi(t)|^2 dt + \frac{(T-T')^2}{4\varepsilon^4\alpha}\|h\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

This concludes the proof taking into account that $\|h\|^2_{L^\infty(0,T;L^2(\Omega))}$ is bounded uniformly for any solution with initial data in a bounded set B by Proposition 13(a) and the term $\int_{T'}^T |\nabla\phi(t)|^2 dt$ is also uniformly bounded if we revise the proof of Proposition 13 since this term appeared (see e.g. (30)), although it was neglected for the posterior calculus. □

5. Compactness of the multi-valued semiflows and attractors

In the above section we have established the existence of absorbing sets for four of the multi-valued semiflows (cf. Corollary 15). Although this does not hold for $((L^2(\Omega))^2 \times L^2(\Omega; [0,1]), \{G(t)\}_{t\geq 0})$ and $(H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega; [0,1]), \{G(t)\}_{t\geq 0})$, we prove that a compactness property holds, whence it is inherited for the rest of semiflows.

Lemma 17. *Under assumption (6), consider any sequence $\{(\phi^n, \theta^n, c^n)\}_n$ of solutions of (1)–(5) with initial data $(\phi_0^n, \theta_0^n, c_0^n)$ and satisfying that $(\phi_0^n, \theta_0^n, c_0^n) \rightharpoonup (\phi_0, \theta_0, c_0)$ weakly in $(L^2(\Omega))^3$. Let us also fix a value $t^* > 0$. Then, $c_0 \in L^2(\Omega; [0,1])$ and there exist a subsequence $\{(\phi^\mu, \theta^\mu, c^\mu)\}_\mu$ and a triplet (ϕ, θ, c), solution of (1)–(5), with initial data (ϕ_0, θ_0, c_0), such that*

(a) *the following convergences hold for all $T > 0$:*

$$\begin{aligned}
(\phi^\mu, \theta^\mu, c^\mu) &\rightharpoonup (\phi, \theta, c) \quad \textit{weakly in} \ \ L^2(0,T; (H^1(\Omega))^3),\\
(\phi_t^\mu, \theta_t^\mu, c_t^\mu) &\rightharpoonup (\phi_t, \theta_t, c_t) \quad \textit{weakly in} \ \ (L^2(0,T; H^1(\Omega)') + L^{4/3}(0,T; L^{4/3}(\Omega)))^2\\
&\qquad\qquad \times L^2(0,T; H^1(\Omega)'),
\end{aligned}$$

(b) $(\phi^\mu(t^*), \theta^\mu(t^*), c^\mu(t^*)) \to (\phi(t^*), \theta(t^*), c(t^*))$ *strongly in* $H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega; [0,1])$.

(c) *If moreover* $\phi_0^n \rightharpoonup \phi_0$ *weakly in* $H^1(\Omega)$, *then one also has for all* $T>0$ *that*

$$\begin{aligned} (\phi^\mu, \theta^\mu, c^\mu) &\rightharpoonup (\phi,\theta,c) \quad \textit{weakly in} \;\; L^2(0,T;H^2(\Omega)\times(H^1(\Omega))^2), \\ (\phi_t^\mu, \theta_t^\mu, c_t^\mu) &\rightharpoonup (\phi_t,\theta_t,c_t) \quad \textit{weakly in} \;\; L^2(0,T;L^2(\Omega)\times(H^1(\Omega)')^2). \end{aligned} \tag{33}$$

Proof. First at all, the fact that $c_0 \in L^2(\Omega;[0,1])$ follows since $L^2(\Omega;[0,1])$ is a convex bounded set of $L^2(\Omega)$, closed for the strong and also for the weak topology.

Now, observe that (a) and (c) are consequences of Theorem 1. More exactly, taking the sequence $\{(\phi^n,\theta^n,c^n)\}_n$ and repeating the estimates in the proof of Theorem 1, one obtains the uniform estimates leading to (a) for general data, and reasoning as in the proof of Proposition 13(b) it yields (c) for more regular data. Indeed, (a) summaries only some of the possible convergences, but in fact (19)–(22) hold. Passing to the limit, one can check analogously to Theorem 1 that (ϕ,θ,c) is a solution with initial data (ϕ_0,θ_0,c_0) (there is no matter with the weak convergence of $(\phi_0^n,\theta_0^n,c_0^n)$).

So, it only remains to prove (b). Thanks to Proposition 16, after a time less than $T-t^*$, changing eventually data to a new bounded set and time interval, we may restrict to the case of regular initial data, i.e. we can assume that we are in case (c) and (33) holds.

In particular, we also have (for a subsequence, that we relabel the same) that

$$(\phi^\mu,\theta^\mu,c^\mu) \to (\phi,\theta,c) \quad \text{in } L^2(0,T;H^1(\Omega)\times(L^2(\Omega))^2),$$

$$(\phi^\mu(t),\theta^\mu(t),c^\mu(t)) \to (\phi(t),\theta(t),c(t)) \quad \text{in } H^1(\Omega)\times(L^2(\Omega))^2 \; \textit{a.e.} \text{ on } (0,T).$$

On the other hand, we know by Proposition 13 that the sequence $\{(\phi^\mu,\theta^\mu,c^\mu)\}_\mu$ is uniformly bounded in $C([0,T];H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1])$. In particular, it is also bounded in $C([0,T];L^2(\Omega)\times((H^1(\Omega))')^2)$, and as this sequence is equicontinuous there by (33), using the Ascoli-Arzelà Theorem, for a subsequence (relabelled the same)

$$(\phi^\mu(t),\theta^\mu(t),c^\mu(t)) \to (\phi(t),\theta(t),c(t)) \quad \text{in } L^2(\Omega)\times((H^1(\Omega))')^2 \quad \forall t\in[0,T]. \tag{34}$$

Using again the uniform estimates from Proposition 13 for the fixed time t^*, we may extract a weakly converging subsequence (relabelled the same), and we can identify the weak limit thanks to (34),

$$(\phi^\mu(t^*),\theta^\mu(t^*),c^\mu(t^*)) \rightharpoonup (\phi(t^*),\theta(t^*),c(t^*)) \quad \text{weakly in } H^1(\Omega)\times(L^2(\Omega))^2. \tag{35}$$

To obtain (b) we need the convergence of the norm of the involved elements. We proceed similarly to Lemma 4.8 in [8] for each of the three variables.

Namely, what we will apply is that if $J(\cdot)$ and $\{J_\mu(\cdot)\}_\mu$ are continuous and monotone functions on $[0,T]$, and $J_\mu(t)\to J(t)$ a.e. on $[0,T]$, then $J_n(t)\to J(t)$ for all $t\in(0,T)$.

Step 1: Construction of functions J and $\{J_\mu\}_\mu$ for the ϕ variable.

From (26) for initial data in the bounded set B (and the weak limit (ϕ_0,θ_0,c_0), also bounded), we conclude that there exists a constant $C_\phi(B)>0$ such that from (31), neglecting one term in the left hand side,

$$|\nabla\phi^\mu(t)|^2 \le |\nabla\phi^\mu(s)|^2 + C_\phi(B)(t-s) \quad \forall\, 0\le s\le t,\; t\in[0,T],$$

and analogous inequality for ϕ. Therefore we can define the continuous and monotone functions

$$J_{\phi,\mu}(t) = |\nabla\phi^\mu(t)|^2 - C_\phi(B)t, \qquad J_\phi(t) = |\nabla\phi(t)|^2 - C_\phi(B)t.$$

Using the result already announced from [8, Lemma 4.8] we obtain that

$$|\nabla\phi^\mu(t^*)| \to |\nabla\phi(t^*)|.$$

Since we already had the weak convergence (35) and the injection of $H^1(\Omega)$ in $L^2(\Omega)$ is compact, we deduce finally the convergence

$$\phi^\mu(t^*) \to \phi(t^*) \quad \text{in } H^1(\Omega).$$

Step 2: Construction of functions J and $\{J_\mu\}_\mu$ for the θ variable.

We will discuss analogously to the Step 1, but using the variables $u^\mu = C_V\theta^\mu + \frac{l}{2}\phi^\mu$ and $u = C_V\theta + \frac{l}{2}\phi$. If we prove that $\{u^\mu(t^*)\}_\mu$ converges to $u(t^*)$ in $L^2(\Omega)$, by the Step 1, we conclude that $\{\theta^\mu(t^*)\}_\mu$ also converges to $\theta(t^*)$ in $L^2(\Omega)$.

Indeed, this can be done exactly reasoning as before but using now (28), neglecting one term, so that for a suitable constant $C_u(B)$, one has

$$|u^\mu(t)|^2 \le |u^\mu(s)|^2 + C_u(B)(t-s) \quad \forall\ 0 \le s \le t,\ t \in [0,T],$$

and analogous inequality for u. Therefore we can define again continuous and monotone functions

$$J_{u,\mu}(t) = |u^n(t)|^2 - C_u(B)t, \qquad J_u(t) = |u(t)|^2 - C_u(B)t$$

and proceed as before.

Step 3: Construction of functions J and $\{J_\mu\}_\mu$ for the c variable.

The argument will be similar to the above cases. However, observe that in Proposition 13 we did not analyze the behaviour of c as far as it is always bounded in $[0,1]$. It is easy to obtain a similar inequality to (28) or (31) as follows. Taking c as a test function in (9), and using the fact that $c \in [0,1]$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|c|^2 + K_2|\nabla c|^2 &= -K_2M\int_\Omega (c(1-c)\nabla\phi\cdot\nabla c)dx \\
&\le K_2M|\nabla c||\nabla\phi| \\
&\le \frac{K_2}{2}|\nabla c|^2 + \frac{K_2M^2}{2}|\nabla\phi|^2,
\end{aligned}$$

whence

$$\frac{d}{dt}|c|^2 + K_2|\nabla c|^2 \le K_2M^2|\nabla\phi|^2. \tag{36}$$

Now, for an adequate constant $C_c(B) > 0$, inequality (36) applied to the different solutions yields

$$|c^n(t)|^2 \le |c^n(s)|^2 + C_c(B)(t-s) \quad \forall\ 0 \le s \le t,\ t \in [0,T],$$

and analogous inequality for c. Therefore we can define again continuous and monotone functions

$$J_{c,\mu}(t) = |c^\mu(t)|^2 - C_c(B)t, \qquad J_c(t) = |c(t)|^2 - C_c(B)t,$$

and conclude, as before, that $c^n(t^*) \to c(t^*)$ in $L^2(\Omega)$. □

Remark 18. *The statement (b) above is formulated for t^* because we need $t^* \in (0,T)$ in the argument of* [8]. *Nevertheless, observe that both, t^* and T, are arbitrary.*

A direct consequence from the above result is the following

Corollary 19. *Assume that (6) holds. Then, all semiflows associated to problem (1)–(5) given in Proposition 11, are compact, i.e. for any $T>0$, the application $G(T,\cdot)$ maps bounded onto relatively compact sets (in their respective metric).*

Proof. The claim for $(H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1]),\{G(t)\}_{t\geq 0})$ holds by Lemma 17. For $((L^2(\Omega))^2\times L^2(\Omega;[0,1]),\{G(t)\}_{t\geq 0})$, the compactness follows from Proposition 16 and Lemma 17.

The rest are (an inherited) consequence of the above ones and Proposition 3. □

Remark 20. *In fact, as a consequence of Lemma 17 one has a stronger result than the above corollary, since those semiflows are compact not only in their own phase-spaces, but the set $G(T,B)$ is relatively compact in $H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1])$.*

Corollary 21. *Under assumption (6), the multi-valued semiflows $(X,\{G(t)\}_{t\geq 0})$, where X can be $H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1])$, $(L^2(\Omega))^2\times L^2(\Omega;[0,1])$, L^2_γ, H^1_γ, for any $\gamma\in\mathbb{R}$, $\mathcal{L}^2_\rho$, or $\mathcal{H}^1_\rho$ for any $\rho\in\mathbb{R}_+$, possess the following properties:*

(a) *it has compact values, i.e. $G:\mathbb{R}_+\times X\to K(X)$,*
(b) *for each fixed $t\geq 0$, $G(t,\cdot):X\to K(X)$ is upper semicontinuous.*

Proof. Claim (a) is obvious applying Lemma 17 to B a singleton. Indeed, if $t>0$, from Remark 20 we know that $G(t,x)$ is compact in $H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1])$.

The claim (b) follows by a contradiction argument. Denote X any of the above metric spaces. Then, there should exist $(\phi_0,\theta_0,c_0)\in X$, a neighbourhood M of $G(t,\phi_0,\theta_0,c_0)$, and a sequence $\{(\phi_0^n,\theta_0^n,c_0^n)\}_n$ with $\lim_{n\to+\infty}(\phi_0^n,\theta_0^n,c_0^n)=(\phi_0,\theta_0,c_0)$ in X, such that solutions $(\phi^n,\theta^n,c^n)\in D(\phi_0^n,\theta_0^n,c_0^n)$ satisfy that

$$(\phi^n(t),\theta^n(t),c^n(t))\notin M\quad\forall n\in\mathbb{N}.$$

But this is a contradiction since we can fix $t^*=t<T$ and extract a subsequence $\{(\phi^\mu,\theta^\mu,c^\mu)\}_\mu$ converging to a solution $(\phi,\theta,c)\in D(\phi_0,\theta_0,c_0)$ and satisfying (b) in Lemma 17. □

As a consequence of the above results, we are able to establish our main result.

Theorem 22. *Under assumption (6), the multi-valued semiflows $(\mathcal{L}^2_\rho,\{G(t)\}_{t\geq 0})$ and $(\mathcal{H}^1_\rho,\{G(t)\}_{t\geq 0})$, for any $\rho\in\mathbb{R}_+$, possesses global attractors $\mathcal{A}_{\mathcal{L}^2_\rho}$ and $\mathcal{A}_{\mathcal{H}^1_\rho}$ respectively. Moreover, it holds*

$$\mathcal{A}_{\mathcal{H}^1_\rho}=\mathcal{A}_{\mathcal{L}^2_\rho}\quad\forall\rho\in\mathbb{R}_+. \tag{37}$$

Proof. The existence of attractors is a consequence of Theorem 7 and Remark 9 applied to both semiflows, since the sufficient conditions hold from corollaries 15, 19 and 21.

In order to prove (37), consider a fixed value $\rho\in\mathbb{R}_+$.

Since $\mathcal{A}_{\mathcal{L}^2_\rho}$ is compact in $\mathcal{L}^2_\rho$, in particular is bounded. By Proposition 16, the set $G(T,\mathcal{A}_{\mathcal{L}^2_\rho})$ is bounded in $\mathcal{H}^1_\rho$. Using that

$$\lim_{t\to+\infty}\operatorname{dist}_{\mathcal{H}^1_\rho}(G(t,\mathcal{A}_{\mathcal{L}^2_\rho}),\mathcal{A}_{\mathcal{H}^1_\rho})=0,$$

and the invariance of $\mathcal{A}_{\mathcal{L}^2_\rho}$ for G, we deduce that $\mathcal{A}_{\mathcal{L}^2_\rho}\subset\mathcal{A}_{\mathcal{H}^1_\rho}$. The other inclusion is easier, since $\mathcal{H}^1_\rho\subset\mathcal{L}^2_\rho$, and therefore

$$\lim_{t\to+\infty}\operatorname{dist}_{\mathcal{L}^2_\rho}(G(t,\mathcal{A}_{\mathcal{H}^1_\rho}),\mathcal{A}_{\mathcal{L}^2_\rho})=0,$$

but we have that $G(t, \mathcal{A}_{\mathcal{H}^1_\rho}) = \mathcal{A}_{\mathcal{H}^1_\rho}$ for all t, again by the invariance of the attractor. This lead analogously to $\mathcal{A}_{\mathcal{H}^1_\rho} \subset \mathcal{A}_{\mathcal{L}^2_\rho}$. □

Corollary 23. *Under assumption (6), the multi-valued semiflows* $(L^2_\gamma, \{G(t)\}_{t\geq 0})$ *and* $(H^1_\gamma, \{G(t)\}_{t\geq 0})$ *also have global attractors* $\mathcal{A}_{L^2_\gamma}$ *and* $\mathcal{A}_{H^1_\gamma}$ *for any* $\gamma \in \mathbb{R}$. *Moreover, the following equalities hold:*

$$\mathcal{A}_{L^2_\gamma} = \mathcal{A}_{H^1_\gamma}, \quad \forall \gamma \in \mathbb{R}, \tag{38}$$

$$\bigcup_{|\gamma|\leq\rho} \mathcal{A}_{L^2_\gamma} = \mathcal{A}_{\mathcal{L}^2_\rho}, \quad \forall \rho \in \mathbb{R}_+, \tag{39}$$

$$\bigcup_{|\gamma|\leq\rho} \mathcal{A}_{H^1_\gamma} = \mathcal{A}_{\mathcal{H}^1_\rho}, \quad \forall \rho \in \mathbb{R}_+. \tag{40}$$

Proof. The existence of the global attractors $\mathcal{A}_{L^2_\gamma}$ and $\mathcal{A}_{H^1_\gamma}$ for their respective multi-valued semiflows and the relation (38) follow analogously to Theorem 22.

Secondly, the equality (39) can be proved in two steps. The inclusion to the right is a consequence of the relation $L^2_\gamma \subset \mathcal{L}^2_\rho$ for $|\gamma| \leq \rho$, and the well-known fact that the global attractor is for its semiflow the biggest compact invariant set. The inclusion to the left follows from Proposition 3, the compactness and invariance of the set $\mathcal{A}_{\mathcal{L}^2_\rho} \cap L^2_\gamma$, and the same well-known fact cited above.

Finally, the relation (40) follows analogously to the previous case. □

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(Pedro Marín-Rubio and José Real) Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla (Spain)
E-mail address, Pedro Marín-Rubio: `pmr@us.es`
E-mail address, José Real: `jreal@us.es`

(Gabriela Planas) Departamento de Matemática, Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Caixa Postal 6065, 13083-859 Campinas - SP, Brazil
E-mail address, G. Planas: `gplanas@ime.unicamp.br`